

## INVARIANT FUNCTIONS ON GRASSMANNIANS

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*This article is dedicated to Professor Sigurdur Helgason on the occasion of his 80th birthday*

**ABSTRACT.** It is known, that every function on the unit sphere in  $\mathbb{R}^n$ , which is invariant under rotations about some coordinate axis, is completely determined by a function of one variable. Similar results, when invariance of a function reduces dimension of its actual argument, hold for every compact symmetric space and can be obtained in the framework of Lie-theoretic consideration. In the present article, this phenomenon is given precise meaning for functions on the Grassmann manifold  $G_{n,i}$  of  $i$ -dimensional subspaces of  $\mathbb{R}^n$ , which are invariant under orthogonal transformations preserving complementary coordinate subspaces of arbitrary fixed dimension. The corresponding integral formulas are obtained. Our method relies on bi-Stiefel decomposition and does not invoke Lie theory.

## INTRODUCTION

Integral formulas for semisimple Lie groups and related symmetric spaces constitute a core of geometric analysis. Many such formulas are presented in remarkable books by S. Helgason [H94]-[H01]. They are intimately connected with decompositions of the corresponding Lie algebras and Haar measures, and amount to pioneering works by H. Weyl, É. Cartan, Harish-Chandra; see bibliographical notes in [H00, p. 231]. One of such important formulas is related to the Cartan decomposition  $G = KAK$ . Its generalization  $G = KAH$  is due to Flensted-Jensen [FJ2] in the noncompact case and Hoogenboom [Ho1, Ho2] for  $G$  compact.

To be more specific, let  $U$  be a connected compact real semisimple Lie group. Let  $\theta$  and  $\sigma$  be two commuting involutions of  $U$ ,

$$U^\theta = \{u \in U \mid \theta(u) = u\} \quad (\text{similarly for } U^\sigma).$$

Let  $K$  and  $H$  be closed subgroups of  $U$  such that

$$(0.1) \quad (U^\theta)_0 \subseteq K \subseteq U^\theta \quad \text{and} \quad (U^\sigma)_0 \subseteq H \subseteq U^\sigma,$$

where the subscript  $_0$  denotes the corresponding connected component of the unity  $e$ . Subgroups, which obey (0.1), are called symmetric and the quotient spaces  $U/K$  and  $U/H$  are compact symmetric spaces. Our interest will be in the double coset space  $K \backslash U/H$ .

For the Lie algebra  $\mathfrak{u}$  of  $U$ , we consider two Cartan decompositions  $\mathfrak{u} = \mathfrak{k} + \mathfrak{p}$  and  $\mathfrak{u} = \mathfrak{h} + \mathfrak{q}$ , where  $\mathfrak{p}$  and  $\mathfrak{q}$  are  $-1$  eigenspaces of differentials  $d\theta$  and  $d\sigma$  in  $\mathfrak{u}$ ,

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respectively. Let  $\mathfrak{a}$  be a maximal abelian subalgebra in  $\mathfrak{p} \cap \mathfrak{q}$ ,  $A = \exp(\mathfrak{a})$ . Let  $M = Z_{K \cap H}(A)$  denote the centralizer of  $A$  in  $K \cap H$ . According to [Ho1, formula (4.12)], there is a nonnegative function  $\delta$  on  $A$ , that can be expressed in terms of sin and cos functions, the restricted roots of  $\mathfrak{a}_{\mathbb{C}}$  in  $\mathfrak{u}_{\mathbb{C}}$ , and the multiplicities, so that

$$(0.2) \quad \int_{U/H} f(uH) duH = c \int_A \int_{K/M} f(kaH) \delta(a) dkM da, \quad f \in C(U/H).$$

The constant  $c$  can be explicitly evaluated. Here, as elsewhere in this article, except where clearly stated, the invariant measure on a compact group is normalized to be one.

Formula (0.2) is a consequence of the corresponding decomposition of the Haar measure  $du$  on  $U$ ; see [Ho1, Theorem 4.7]. Fundamental results in this directions for noncompact Lie groups were first obtained by Berger [Be]; for a modern account, see Flensted-Jensen [FJ1, FJ2], [Ma]. The method by Hoogenboom [Ho1] gives integral formulas both for the noncompact and compact cases. If  $f$  is left  $K$ -invariant then (0.2) yields

$$(0.3) \quad \int_{U/H} f(uH) duH = c \int_A f_0(a) \delta(a) da$$

for some function  $f_0$  on  $A$ . The map  $f \rightarrow f_0$  preserves the smoothness (or integrability) of  $f$  up to the weight function  $\delta$ . Formula (0.3) can be applied to the study of left  $K$ -invariant functions  $f$  on the symmetric space  $U/H$ .

In the present article, we obtain explicit characterization of such functions, when  $U$  stands for the orthogonal group  $O(n)$ ,  $U/H$  is the Grassmann manifold

$$G_{n,i} = O(n)/(O(n-i) \times O(i)) = SO(n)/S(O(n) \times O(n-i))$$

of  $i$ -dimensional subspaces of  $\mathbb{R}^n$ , and the subgroup  $K$  has the form

$$(0.4) \quad K \equiv K_\ell = \left\{ \gamma \in O(n) \mid \gamma = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad \alpha \in O(n-\ell), \beta \in O(\ell) \right\} \\ \sim O(n-\ell) \times O(\ell).$$

In this setting,  $i$  and  $\ell$  are arbitrary integers,  $1 \leq i, \ell \leq n-1$ . Note that  $U = O(n)$ ,  $H = O(n-i) \times O(i)$ , and  $K = O(n-\ell) \times O(\ell)$  are not connected, but one can show that (0.2) and (0.3) are still valid. The subgroup  $K_\ell$  is symmetric in the sense that  $K_\ell = O(n)^{\theta_\ell}$ , where the involution  $\theta_\ell$  is defined by

$$\theta_\ell(u) = I_{n-\ell,\ell} u I_{n-\ell,\ell}; \quad u \in O(n), \quad I_{n-\ell,\ell} = \begin{bmatrix} I_{n-\ell} & 0 \\ 0 & -I_\ell \end{bmatrix}.$$

In fact, one can readily see that

$$\theta_\ell \left( \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right) = \begin{bmatrix} A & -B \\ -C & D \end{bmatrix}.$$

Since  $I_{n-\ell,\ell}$  and  $I_{n-i,i}$  commute, the involutions  $\theta_\ell$  and  $\theta_i$  commute too, and the results by Hoogenboom can be applied.

Unlike the Lie theoretic argument sketched above, our consideration does not invoke the Lie-theoretic techniques and is motivated by application to problems in convex geometry, dealing with sections of star-shaped bodies with symmetries; see, e.g., [R]. We also recall, that if one would like to use the representation of noncompact semisimple Lie groups, in particular the principal series representations

and intertwining operators, then  $G_{n,i}$  can also be written as  $\mathrm{SL}(n, \mathbb{R})/P_i$ , where  $P_i$  is the parabolic subgroup

$$P_i = \left\{ \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \in \mathrm{SL}(n, \mathbb{R}) \mid A \in \mathrm{GL}(i, \mathbb{R}), B \in \mathrm{GL}(n-i, \mathbb{R}), X \in \mathfrak{M}_{i, n-i} \right\}$$

and  $\mathfrak{M}_{r,s} \simeq \mathbb{R}^{rs}$  stands for the space of  $r \times s$  real matrices.

Let us explain the idea of the paper by the simple example

$$S^{n-1} = \mathrm{SO}(n)/\mathrm{SO}(n-1) = \mathrm{O}(n)/\mathrm{O}(n-1),$$

where  $S^{n-1}$  is the unit sphere in  $\mathbb{R}^n$  with the area  $\sigma_{n-1} = 2\pi^{n/2}/\Gamma(n/2)$ . We fix an integer  $\ell$ ,  $1 \leq \ell \leq n-1$ , and write

$$(0.5) \quad \mathbb{R}^n = \mathbb{R}^{n-\ell} \oplus \mathbb{R}^\ell, \quad \mathbb{R}^{n-\ell} = \bigoplus_{j=1}^{n-\ell} \mathbb{R}e_j, \quad \mathbb{R}^\ell = \bigoplus_{j=n-\ell+1}^n \mathbb{R}e_j,$$

where  $e_1, e_2, \dots, e_n$  are the coordinate unit vectors. According to (0.5), every point  $\theta \in S^{n-1}$  can be represented in bi-spherical coordinates as

$$(0.6) \quad \theta = \begin{bmatrix} u \sin \omega \\ v \cos \omega \end{bmatrix}, \quad u \in S^{n-\ell-1}, \quad v \in S^{\ell-1}, \quad 0 \leq \omega \leq \frac{\pi}{2},$$

so that  $d\theta = \sin^{n-\ell-1} \omega \cos^{\ell-1} \omega du dv d\omega$ , where  $d\theta, du$ , and  $dv$  denote the relevant (non-normalized) volume elements; see, e.g., [VK]. Clearly,  $\cos^2 \omega = \theta^t \sigma \sigma^t \theta = \theta^t \mathrm{Pr}_{\mathbb{R}^\ell} \theta$ , where  $\mathrm{Pr}_{\mathbb{R}^\ell}$  denotes the orthogonal projection onto  $\mathbb{R}^\ell$  and

$$(0.7) \quad \sigma = [e_{n-\ell+1}, \dots, e_n] = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix}.$$

The following statement is an immediate consequence of (0.6).

**Theorem 1.** *For  $s \in [0, 1]$ , let*

$$d\nu(s) = s^{\ell/2-1} (1-s)^{(n-\ell)/2-1} ds.$$

*An integrable function  $f$  on  $S^{n-1}$  is  $K_\ell$ -invariant if and only if there is a function  $f_0 \in L^1([0, 1]; d\nu)$  such that  $f(\theta) = f_0(s)$ , where  $s^{1/2} = (\theta^t \mathrm{Pr}_{\mathbb{R}^\ell} \theta)^{1/2}$  is the cosine of the angle between the unit vector  $\theta$  and the coordinate subspace  $\mathbb{R}^\ell$ . Moreover,*

$$(0.8) \quad \begin{aligned} \int_{S^{n-1}} f(\theta) d\theta &= c \int_0^{\pi/2} f_0(\cos^2 \omega) \sin^{n-\ell-1} \omega \cos^{\ell-1} \omega d\omega \\ &= \frac{c}{2} \int_0^1 f_0(s) d\nu(s), \quad c = \sigma_{\ell-1} \sigma_{n-\ell-1}. \end{aligned}$$

## 1. MAIN RESULTS

Let  $G_{n,i}$  be the Grassmann manifold of  $i$ -dimensional linear subspaces  $\xi$  of  $\mathbb{R}^n$ ,  $1 \leq i \leq n-1$ . It is assumed, that  $G_{n,i}$  is endowed with the  $\mathrm{O}(n)$ -invariant measure  $d\xi$  of total mass 1. For  $1 \leq \ell \leq n-1$  let  $m = \min\{i, \ell\}$ . We will need the simplex

$$(1.1) \quad \Lambda_m = \{\lambda = (\lambda_1, \dots, \lambda_m) \mid 1 \geq \lambda_1 \geq \dots \geq \lambda_m \geq 0\},$$

and the Siegel gamma function

$$(1.2) \quad \Gamma_m(\alpha) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(\alpha - j/2).$$

To every subspace  $\xi \in G_{n,i}$ , we assign a point  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$  in  $\Lambda_m$ , so that  $\lambda_1, \dots, \lambda_m$  are eigenvalues of the positive semi-definite matrix

$$(1.3) \quad r = \begin{cases} \Theta^t \text{Pr}_{\mathbb{R}^\ell} \Theta & \text{if } i \leq \ell, \\ \Psi^t \text{Pr}_\xi \Psi & \text{if } i > \ell. \end{cases}$$

Here  $\Theta = (\theta_{i,j})_{n \times i}$  and  $\Psi = (\psi_{i,j})_{n \times \ell}$  are arbitrary fixed matrices whose columns form an orthonormal basis in  $\xi$  and  $\mathbb{R}^\ell$ , respectively;  $\Theta^t$  and  $\Psi^t$  are the corresponding transposed matrices;  $\text{Pr}_\xi$  and  $\text{Pr}_{\mathbb{R}^\ell}$  denote the relevant orthogonal projections. Clearly,  $\boldsymbol{\lambda}$  is independent of the choice of orthonormal bases in  $\xi$  and  $\mathbb{R}^\ell$ .

**Theorem 2.** *Assume that  $1 \leq i, \ell \leq n-1$  are such that  $i + \ell \leq n$ . Let  $m = \min\{i, \ell\}$ . For  $\boldsymbol{\lambda} \in \Lambda_m$ , we set*

$$d\nu(\boldsymbol{\lambda}) = \prod_{1 \leq j < k \leq m} (\lambda_j - \lambda_k) \prod_{j=1}^m \lambda_j^\alpha (1 - \lambda_j)^\beta d\lambda_j,$$

$$\alpha = (n - \ell - i - 1)/2, \quad \beta = (|\ell - i| - 1)/2.$$

*An integrable function  $f$  on  $G_{n,i}$  is  $K_\ell$ -invariant if and only if there is a function  $f_0 \in L^1(\Lambda_m; d\nu)$  such that  $f(\xi) = f_0(\boldsymbol{\lambda})$ , where  $\boldsymbol{\lambda}$  is formed by eigenvalues of matrix (1.3). Moreover,*

$$(1.4) \quad \int_{G_{n,i}} f(\xi) d\xi = c \int_{\Lambda_m} f_0(\boldsymbol{\lambda}) d\nu(\boldsymbol{\lambda}),$$

where

$$(1.5) \quad c = c_m \begin{cases} \Gamma_i(n/2)/\Gamma_i(\ell/2)\Gamma_i((n-\ell)/2) & \text{if } i \leq \ell, \\ \Gamma_\ell(n/2)/\Gamma_\ell(i/2)\Gamma_\ell((n-i)/2) & \text{if } i \geq \ell, \end{cases}$$

$$c_m = \pi^{(m^2+m)/4} \left( \prod_{j=1}^m j \Gamma(j/2) \right)^{-1}.$$

The geometrical meaning of  $\lambda_1, \dots, \lambda_m$  in the equality

$$f(\xi) = f_0(\boldsymbol{\lambda}) \equiv f_0(\lambda_1, \dots, \lambda_m)$$

is that  $\lambda_1 = \cos^2 \omega_1, \dots, \lambda_m = \cos^2 \omega_m$ , where  $\omega_1, \dots, \omega_m$  are canonical angles, which determine the relative position of a subspace  $\xi \in G_{n,i}$  with respect to the coordinate subspace  $\mathbb{R}^\ell$ ; see, e.g., [C, J].

The proof of Theorem 2 relies on the bi-Stiefel decomposition of the Haar measure on the Stiefel manifold [GR, Herz]; see Lemma 5. A simple proof of it, presented in Section 2, is an adaptation of the argument of Zhang [Zh] to the real case.

We conjecture, that our method extends to the hyperbolic case  $\mathbb{H}^n$ , when  $i$ -dimensional planes are substituted by  $i$ -dimensional totally geodesic submanifolds of  $\mathbb{H}^n$ . We plan to study this case in the context of related problems of integral geometry in forthcoming publications.

## 2. THE STIEFEL MANIFOLD AND MORE NOTATION

As before,  $\mathfrak{M}_{n,m} \simeq \mathbb{R}^{nm}$  denotes the space of real matrices  $x = (x_{i,j})$  having  $n$  rows and  $m$  columns;  $dx = \prod_{i=1}^n \prod_{j=1}^m dx_{i,j}$  is the volume element on  $M_{n,m}$ . Given a square matrix  $a$ , let  $|a| := |\det(a)|$ . Let  $\mathcal{S}_m \simeq \mathbb{R}^{m(m+1)/2}$  be the space of  $m \times m$  real symmetric matrices  $s = (s_{i,j})$  with the volume element  $ds = \prod_{i \leq j} ds_{i,j}$ . We denote by  $\mathcal{P}_m \subset \mathcal{S}_m$  the cone of positive definite matrices in  $\mathcal{S}_m$ . Given  $a$  and  $b$  in  $\mathcal{S}_m$ , the symbol  $\int_a^b f(s) ds$  denotes the integral over the compact set  $(a + \mathcal{P}_m) \cap (b - \mathcal{P}_m)$  and  $\int_a^\infty f(s) ds$  means the integral over  $a + \mathcal{P}_m$ . The group  $G = \text{GL}(m, \mathbb{R})$  acts transitively on  $\mathcal{P}_m$  by the rule  $g \cdot r := grg^t$ ,  $g \in G$ . The corresponding  $G$ -invariant measure on  $\mathcal{P}_m$  is

$$(2.1) \quad d_* r = |r|^{-d} dr, \quad d = (m+1)/2,$$

[T, p. 18]. For  $n \geq m$ , let  $V_{n,m} = \{v \in \mathfrak{M}_{n,m} \mid v^t v = I_m\}$  be the Stiefel manifold of orthonormal  $m$ -frames in  $\mathbb{R}^n$ . The group  $O(n)$  acts transitively on  $V_{n,m}$  by the rule  $\gamma : v \rightarrow \gamma v$ ,  $\gamma \in O(n)$ , in the sense of matrix multiplication. Let

$$(2.2) \quad \sigma_m = \begin{bmatrix} 0 \\ I_m \end{bmatrix}.$$

Most of the time we simply write  $\sigma$  for  $\sigma_m$ . The stabilizer of  $\sigma$  is

$$O(n-m) \simeq \left\{ \begin{bmatrix} A & 0 \\ 0 & I_m \end{bmatrix} \mid A \in O(n-m) \right\}.$$

Hence  $V_{n,m} = O(n)/O(n-m)$ . We fix the corresponding  $O(n)$ -invariant measure  $dv$  on  $V_{n,m}$  so that

$$(2.3) \quad \int_{V_{n,m}} dv = \sigma_{n,m} = \frac{2^m \pi^{nm/2}}{\Gamma_m(n/2)},$$

$\Gamma_m(\cdot)$  being the Siegel gamma function (1.2); see [Mu, p. 70], [J, p. 57], [FK, p. 351]. The measure  $dv$  is also right  $O(m)$ -invariant.

Let

$$\mathfrak{M}_{n,m}^* = \{x \in \mathfrak{M}_{n,m} \mid \text{rank}(x) = m\}.$$

Then  $\mathfrak{M}_{n,m}^*$  is open, dense and of full measure in  $\mathfrak{M}_{n,m}$ . Define

$$(2.4) \quad \varphi : V_{n,m} \times \mathcal{P}_m \rightarrow \mathfrak{M}_{n,m}^*, \quad (v, r) \mapsto x = vr^{1/2}.$$

Then  $\varphi$  is surjective and  $r = x^t x \in \mathcal{P}_m$  depends smoothly on  $x$ . The following lemma describes the measure  $dx$  on  $\mathfrak{M}_{n,m}^*$  in terms of  $V_{n,m}$  and  $\mathcal{P}_m$ .

**Lemma 3.** *Assume that  $n \geq m$ . Let the notation be as above. Then*

$$dx = 2^{-m} |r|^{n/2} d_* r dv.$$

The polar decomposition (2.4) can be found in many sources, e.g., in [Herz, p. 482], [GK1, p. 93], [Mu, pp. 66, 591], [FT, p. 130].

The next statement, which is actually due to Zhang [Zh], contains a higher-rank generalization of the polar decomposition of the Lebesgue measure in the quarter-plane.

**Lemma 4.** *Let  $F$  be a function on  $\mathcal{P}_m \times \mathcal{P}_m$ ,  $d = (m + 1)/2$ . Then*

$$(2.5) \quad \begin{aligned} & \int_{\mathcal{P}_m \times \mathcal{P}_m} F(p_1, p_2) d_* p_1 d_* p_2 \\ &= \int_0^{I_m} |I_m - r|^{-d} d_* r \int_{\mathcal{P}_m} F(s^{1/2} r s^{1/2}, s^{1/2} (I_m - r) s^{1/2}) d_* s \end{aligned}$$

*provided that either of these integrals exists in the Lebesgue sense.*

*Proof.*

$$\begin{aligned} \text{l.h.s.} &= \int_{\mathcal{P}_m} d_* p_1 \int_{\mathcal{P}_m} F(p_1, p_1 + p_2 - p_1) |p_2|^{-d} dp_2 \quad (\text{set } p_1 + p_2 = s) \\ &= \int_{\mathcal{P}_m} d_* p_1 \int_{p_1}^{\infty} F(p_1, s - p_1) |s - p_1|^{-d} ds \\ &= \int_{\mathcal{P}_m} ds \int_0^s F(p_1, s - p_1) |s - p_1|^{-d} d_* p_1 \quad (\text{set } p_1 = s^{1/2} r s^{1/2}) \\ &= \int_{\mathcal{P}_m} d_* s \int_0^{I_m} F(s^{1/2} r s^{1/2}, s^{1/2} (I_m - r) s^{1/2}) |I_m - r|^{-d} d_* r, \end{aligned}$$

and (2.5) follows.  $\square$

**Lemma 5.** (bi-Stiefel decomposition) *Let  $k$ ,  $m$ , and  $n$  be positive integers satisfying*

$$1 \leq k \leq n - 1, \quad 1 \leq m \leq n - 1, \quad m \leq \min(k, n - k).$$

*Almost all matrices  $v \in V_{n,m}$  can be represented in the form*

$$(2.6) \quad v = \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_m - r)^{1/2} \end{bmatrix}, \quad u_1 \in V_{n-k,m}, \quad u_2 \in V_{k,m},$$

*so that*

$$(2.7) \quad \int_{V_{n,m}} f(v) dv = \int_0^{I_m} d\mu(r) \int_{V_{n-k,m}} du_1 \int_{V_{k,m}} f \left( \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_m - r)^{1/2} \end{bmatrix} \right) du_2,$$

$$d\mu(r) = 2^{-m} |I_m - r|^{(k-m-1)/2} |r|^{(n-m-k-1)/2} dr.$$

*Proof.* For  $m=1$ , this is a well-known bi-spherical decomposition [VK, pp. 12, 22]. For  $k = m$ , see [Herz, p. 495], where the result was obtained using the Fourier transform technique and Bessel functions of matrix argument. In [GR], Herz's proof was extended to the form presented above and it was conjectured that there is an alternative simple proof that does not need the Fourier transform. Such a proof was given by Zhang [Zh]. For convenience of the reader, we present it here in a slightly different notation.

The result will follow from the bi-polar decomposition of the Lebesgue measure on  $\mathfrak{M}_{n,m}$ . We split  $x \in \mathfrak{M}_{n,m}$  in two blocks, so that  $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ,  $x_1 \in \mathfrak{M}_{n-k,m}$ ,  $x_2 \in \mathfrak{M}_{k,m}$ , and write each block in polar coordinates according to Lemma 3. This

gives

$$(2.8) \quad \begin{aligned} \int_{\mathfrak{M}_{n,m}} f(x) dx &= \int_{V_{n-k,m}} du_1 \int_{V_{k,m}} du_2 \\ &\times \int_{\mathcal{P}_m \times \mathcal{P}_m} f \left( \begin{bmatrix} u_1 p_1^{1/2} \\ u_2 p_2^{1/2} \end{bmatrix} \right) h(p_1, p_2) d_* p_1 d_* p_2, \end{aligned}$$

$$(2.9) \quad h(p_1, p_2) = 2^{-2m} |p_1|^{(n-k)/2} |p_2|^{k/2}.$$

By (2.5), the integral over  $\mathcal{P}_m \times \mathcal{P}_m$  transforms as

$$(2.10) \quad \int_0^{I_m} dr \int_{\mathcal{P}_m} f \left( \begin{bmatrix} u_1 (s^{1/2} r s^{1/2})^{1/2} \\ u_2 (s^{1/2} (I_m - r) s^{1/2})^{1/2} \end{bmatrix} \right) \tilde{h}(r, s) d_* s,$$

where

$$\tilde{h}(r, s) = 2^{-2m} |I_m - r|^{(k-m-1)/2} |r|^{(n-m-k-1)/2} |s|^{n/2}.$$

Furthermore, one can write

$$(s^{1/2} r s^{1/2})^{1/2} = \gamma_1 r^{1/2} s^{1/2}, \quad (s^{1/2} (I_m - r) s^{1/2})^{1/2} = \gamma_2 (I_m - r)^{1/2} s^{1/2},$$

for some  $\gamma_1, \gamma_2 \in \text{O}(m)$  (just note that  $|\gamma_1| = |\gamma_2| = 1$ ). Hence, changing the order of integration, and using right  $\text{O}(m)$ -invariance of  $du_1$  and  $du_2$ , we easily get

$$(2.11) \quad \begin{aligned} \int_{\mathfrak{M}_{n,m}} f(x) dx &= 2^{-m} \int_{\mathcal{P}_m} |s|^{n/2} d_* s \int_0^{I_m} d\mu(r) \\ &\times \int_{V_{n-k,m}} du_1 \int_{V_{k,m}} f \left( \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_m - r)^{1/2} \end{bmatrix} s^{1/2} \right) du_2, \end{aligned}$$

where  $d\mu(r) = 2^{-m} |I_m - r|^{(k-m-1)/2} |r|^{(n-m-k-1)/2} dr$ . On the other hand, by Lemma 3,

$$(2.12) \quad \int_{\mathfrak{M}_{n,m}} f(x) dx = 2^{-m} \int_{\mathcal{P}_m} |s|^{n/2} d_* s \int_{V_{n,m}} f(v s^{1/2}) dv.$$

Comparing (2.11) and (2.12), we get the result.  $\square$

### 3. PROOF OF THEOREM 2

We divide the proof into two parts.

**Part I.** Let  $i \leq \ell$ , let  $\sigma_i$  be as in (2.2), and let

$$\kappa_i : V_{n,i} = \text{O}(n)/\text{O}(n-i) \rightarrow G_{n,i} = \text{O}(n)/\text{O}(n-i) \times \text{O}(i), \quad g \cdot \sigma_i \rightarrow g \mathbb{R}^i$$

be the canonical map. It is obviously  $\text{O}(n)$ -equivariant. We use this to identify a  $K_\ell$ -invariant function  $f$  on  $G_{n,i}$  with the left  $K_\ell$ -invariant and right  $\text{O}(i)$ -invariant function  $\varphi = f \circ \kappa_i$  on the Stiefel manifold  $V_{n,i}$ . Then

$$(3.1) \quad \int_{G_{n,i}} f(\xi) d\xi = \frac{1}{\sigma_{n,i}} \int_{V_{n,i}} \varphi(\Theta) d\Theta.$$

By Lemma 5 (with  $m = i$ ,  $k = \ell$ ), almost all  $\Theta \in V_{n,i}$  can be represented as

$$(3.2) \quad \Theta = \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_i - r)^{1/2} \end{bmatrix}, \quad u_1 \in V_{n-\ell,i}, \quad u_2 \in V_{\ell,i}, \quad r \in (0, I_i).$$

Since  $\varphi$  is left  $K_\ell$ -invariant, it follows that  $\varphi$  is independent of the bi-Stiefel coordinates  $u$  and  $v$  in (3.2). Thus we can write  $\varphi(\Theta) \equiv \varphi_0(r)$ . By (2.7) and (2.3) we get

$$(3.3) \quad \frac{1}{\sigma_{n,i}} \int_{V_{n,i}} \varphi(\Theta) d\Theta = c \int_0^{I_i} \varphi_0(r) d\nu_1(r),$$

where

$$(3.4) \quad d\nu_1(r) = |I_i - r|^{(\ell-i-1)/2} |r|^{(n-\ell)/2} d_* r,$$

$$(3.5) \quad c = \frac{\sigma_{n-\ell,i} \sigma_{\ell,i}}{2^i \sigma_{n,i}} = \frac{\Gamma_i(n/2)}{\Gamma_i(\ell/2) \Gamma_i((n-\ell)/2)}.$$

Since  $r$  can be expressed through  $\Theta$  as  $r = \Theta^t \sigma \sigma^t \Theta$  and  $\varphi$  is right  $O(i)$ -invariant we obtain

$$\varphi_0(r) = \varphi_0(\Theta^t \sigma \sigma^t \Theta) = \varphi(\Theta) = \varphi(\Theta g) = \varphi_0(\gamma^t \Theta^t \sigma \sigma^t \Theta \gamma) = \varphi_0(\gamma^t r \gamma)$$

for any  $\gamma \in O(i)$ . Hence, if we write  $r$  in polar coordinates

$$(3.6) \quad r = \gamma^t \lambda \gamma, \quad \gamma \in O(i), \quad \lambda = \text{diag}(\lambda_1, \dots, \lambda_i),$$

where  $\lambda_1, \dots, \lambda_i$  are eigenvalues of  $r$ , we get  $\varphi_0(r) = \varphi_0(\lambda)$ . Moreover, using the known formula for the invariant measure

$$(3.7) \quad d_* r = c_i \prod_{1 \leq j < k \leq i} (\lambda_j - \lambda_k) \left( \prod_{j=1}^i \lambda_j^{-(i+1)/2} d\lambda_j \right) d\gamma$$

where

$$c_i = \pi^{(i^2+i)/4} \left( \prod_{j=1}^i j \Gamma(j/2) \right)^{-1},$$

(see [T, p. 23, 43]), we obtain

$$\begin{aligned} \int_0^{I_i} \varphi_0(r) d\nu_1(r) &= c_i \int_{\Lambda_i} \varphi_0(\lambda) \prod_{1 \leq j < k \leq i} (\lambda_j - \lambda_k) \\ &\quad \times \prod_{j=1}^i \lambda_j^{(n-\ell-i-1)/2} (1 - \lambda_j)^{(\ell-i-1)/2} d\lambda_j. \end{aligned}$$

Combining this formula with (3.1) and (3.3), we obtain (1.4).

Conversely, to every function  $f_0$  on  $\Lambda_i$ , we can assign a function  $\varphi_0$  on  $(0, I_i)$  by the rule  $\varphi_0(r) = f_0(\lambda_1, \dots, \lambda_i)$ , where  $\lambda_1, \dots, \lambda_i$  are eigenvalues of  $r$  arranged as  $1 \geq \lambda_1 \geq \dots \geq \lambda_i \geq 0$ . The function  $\varphi_0$  is  $O(i)$ -invariant, i.e.,

$$(3.8) \quad \varphi_0(r) = \varphi_0(g^t r g) \quad \text{for any } g \in O(i),$$

because  $r$  and  $g^t r g$  have the same eigenvalues. Next we define a function  $\varphi$  on  $V_{n,i}$  by  $\varphi(\Theta) = \varphi_0(\Theta^t \sigma \sigma^t \Theta)$ , where  $\sigma = \sigma_\ell \in V_{n,\ell}$ . The function  $\varphi$  is left  $K_\ell$ -invariant, because for any  $\gamma \in K_\ell$  we have  $\gamma^t \sigma \sigma^t \gamma = \sigma \sigma^t$  and, therefore,

$$\varphi(\gamma \Theta) = \varphi_0(\Theta^t \gamma^t \sigma \sigma^t \gamma \Theta) = \varphi_0(\Theta^t \sigma \sigma^t \Theta) = \varphi(\Theta).$$

It remains to note that, owing to (3.8),  $\varphi$  is right  $O(i)$ -invariant and can therefore be identified with a  $K_\ell$ -invariant function on  $G_{n,i}$ .



**Part II.** Let  $i \geq \ell$ . Suppose that  $f$  is a  $K_\ell$ -invariant function on  $G_{n,i}$ , which is identified with a function  $\varphi(\Theta)$  on  $V_{n,i}$  as in Part I. Let  $\Theta = \begin{bmatrix} a \\ b \end{bmatrix}$ ,  $a \in \mathfrak{M}_{n-\ell,i}$ ,  $b = \sigma^t \Theta \in \mathfrak{M}_{\ell,i}$ . Since  $n - \ell \geq i$ , by Lemma 3, we can write  $a$  in polar coordinates as  $a = vr^{1/2}$ ,  $v \in V_{n-\ell,i}$ ,  $r = a^t a = I_i - b^t b$ . Fix any  $v_0$  in  $V_{n-\ell,i}$  and set  $v = \alpha v_0$ ,  $\alpha \in O(n - \ell)$ . Then

$$\Theta = \begin{bmatrix} \alpha v_0 r^{1/2} \\ b \end{bmatrix} = \begin{bmatrix} \alpha & 0 \\ 0 & I_\ell \end{bmatrix} \begin{bmatrix} v_0(I_i - b^t b)^{1/2} \\ b \end{bmatrix}.$$

Since  $\varphi$  is left  $K_\ell$ -invariant, then

$$(3.9) \quad \varphi(\Theta) = \varphi\left(\begin{bmatrix} v_0(I_i - b^t b)^{1/2} \\ b \end{bmatrix}\right) \equiv \varphi_1(b), \quad b = \sigma^t \Theta.$$

We write the transpose of  $b$  in polar coordinates

$$b^t = us^{1/2}, \quad u \in V_{i,\ell}, \quad s = bb^t \in \mathcal{P}_\ell.$$

Then we fix any  $u_0 \in V_{i,\ell}$  and replace  $u$  by  $\beta u_0$  for some  $\beta \in O(i)$ . This gives  $b = s^{1/2} u_0^t \beta^t$  and therefore, since  $\varphi$  is right  $O(i)$ -invariant,

$$\varphi(\Theta) = \varphi(\Theta\beta) = \varphi_1(\sigma^t \Theta\beta) = \varphi_1(b\beta) = \varphi_1(s^{1/2} u_0^t \beta^t \beta) = \varphi_1(s^{1/2} u_0^t).$$

It means that  $\varphi(\Theta)$  is actually a function of  $s$ . Denote it by  $\varphi_0(s)$ . Since  $b = \sigma^t \Theta$  the positive definite matrix  $s = bb^t = \sigma^t \Theta \Theta^t \sigma = \sigma^t \text{Pr}_\xi \sigma$  lies in the “interval”  $(0, I_\ell)$ . Here  $\text{Pr}_\xi \sigma$  denotes the orthogonal projection of  $\sigma \in V_{n,\ell}$  onto the subspace  $\xi \in G_{n,i}$ . Thus

$$f(\xi) \equiv \varphi(\Theta) = \varphi_0(\sigma^t \Theta \Theta^t \sigma) = \varphi_0(\sigma^t \text{Pr}_\xi \sigma).$$

Since  $\varphi$  is left-invariant under left translation by  $\tilde{\beta} = \begin{bmatrix} I_{n-\ell} & 0 \\ 0 & \beta \end{bmatrix}$ ,  $\beta \in O(\ell)$ , i.e.,  $f(\tilde{\beta}\xi) = f(\xi)$ , it follows that  $\varphi_0(\sigma^t \text{Pr}_{\tilde{\beta}\xi} \sigma) = \varphi_0(\sigma^t \text{Pr}_\xi \sigma)$ , and therefore

$$(3.10) \quad \varphi_0(s) = \varphi_0(\sigma^t \text{Pr}_\xi \sigma) = \varphi_0(\sigma^t \text{Pr}_{\tilde{\beta}\xi} \sigma) = \varphi_0(\sigma^t \tilde{\beta} \Theta \Theta^t \tilde{\beta}^t \sigma).$$

Since

$$\tilde{\beta}^t \sigma = \begin{bmatrix} I_{n-\ell} & 0 \\ 0 & \beta^t \end{bmatrix} \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} = \begin{bmatrix} 0 \\ \beta^t \end{bmatrix} = \begin{bmatrix} 0 \\ I_\ell \end{bmatrix} \beta^t = \sigma \beta^t,$$

equation (3.10) implies that for all  $\beta \in O(\ell)$  we have

$$(3.11) \quad \varphi_0(s) = \varphi_0(\beta \sigma^t \Theta \Theta^t \sigma \beta^t) = \varphi_0(\beta s \beta^t).$$

Thus,  $\varphi_0$  depends only on the eigenvalues  $\lambda_1, \dots, \lambda_\ell$  of  $s = \sigma^t \text{Pr}_\xi \sigma$ ,  $\varphi_0(s) = \varphi_0(\text{diag}(\lambda_1, \dots, \lambda_\ell))$ . Finally, we write

$$f(\xi) = \varphi_0(\text{diag}(\lambda_1, \dots, \lambda_\ell)) \equiv f_0(\lambda_1, \dots, \lambda_\ell).$$

For the corresponding integral formula we have

$$\begin{aligned}
I &= \int_{G_{n,i}} f(\xi) d\xi = \frac{1}{\sigma_{n,i}} \int_{V_{n,i}} \varphi(\Theta) d\Theta \\
&= \frac{1}{\sigma_{n,i}} \int_{V_{n,i}} \varphi_0(\sigma^t \Theta \Theta^t \sigma) d\Theta \\
&\quad (\text{replace } \Theta \text{ by } g^t \sigma_i, g \in O(n), \quad \sigma_i = \begin{bmatrix} 0 \\ I_i \end{bmatrix} \in V_{n,i}) \\
&= \int_{O(n)} \varphi_0(\sigma^t g^t \sigma_i \sigma_i^t g \sigma) dg = \frac{1}{\sigma_{n,\ell}} \int_{V_{n,\ell}} \varphi_0(\Psi^t \sigma_i \sigma_i^t \Psi) d\Psi.
\end{aligned}$$

The last integral can be written in bi-Stiefel coordinates by setting

$$\Psi = \begin{bmatrix} u_1 r^{1/2} \\ u_2 (I_\ell - r)^{1/2} \end{bmatrix}, \quad u_1 \in V_{n-i,\ell}, \quad u_2 \in V_{i,\ell}, \quad r = \Psi^t \sigma_i \sigma_i^t \Psi.$$

Then Lemma 5 gives

$$(3.12) \quad I = c \int_0^{I_\ell} \varphi_0(r) d\nu_2(r),$$

where

$$(3.13) \quad d\nu_2(r) = |I_\ell - r|^{(i-\ell-1)/2} |r|^{(n-\ell-i-1)/2} dr,$$

$$(3.14) \quad c = \frac{\sigma_{n-i,\ell} \sigma_{i,\ell}}{2^\ell \sigma_{n,\ell}} = \frac{\Gamma_\ell(n/2)}{\Gamma_\ell(i/2) \Gamma_\ell((n-i)/2)}.$$

Since  $\varphi_0$  is  $O(\ell)$ -invariant (see (3.11)), we can write (3.12) in polar coordinates in the required form (1.4).

Conversely, as in Part I, every function  $f_0$  on  $\Lambda_\ell$  can be associated with an  $O(\ell)$ -invariant function  $\varphi_0$  on  $(0, I_\ell)$ , and the latter generates a function  $\varphi(\Theta) = \varphi_0(\sigma^t \Theta \Theta^t \sigma)$ . This function is left  $K_\ell$ -invariant on  $V_{n,i}$ . Indeed, let  $\gamma = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} \in K_\ell$ ,  $\alpha \in O(n-\ell)$ ,  $\beta \in O(\ell)$ . Then  $\varphi(\gamma\Theta) = \varphi_0(\sigma^t \gamma \Theta \Theta^t \gamma^t \sigma)$ . Since

$$\sigma^t \gamma = [0, I_\ell] \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix} = [0, \beta] = \beta \sigma^t,$$

then  $\varphi(\gamma\Theta) = \varphi_0(\beta \sigma^t \Theta \Theta^t \sigma \beta^t) = \varphi_0(\sigma^t \Theta \Theta^t \sigma) = \varphi(\Theta)$ . Furthermore,  $\varphi(\Theta)$  is obviously right  $O(i)$ -invariant, and therefore, can be identified with a  $K_\ell$ -invariant function on  $G_{n,i}$ .

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